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Control of Linear Stochastic Time Delayed Systems

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The linear-quadratic control problem of stochastic time-delayed systems has been solved using function space method. The solution demonstrates directly that the "separation theorem" holds for such systems.

1. INTRODUCTION

The optimal control problem for linear time-delayed systems with quadratic cost functional has been studied by various methods. Manitius [1] contains a comprehensive list of references. In this paper, we solve the corresponding stochastic control problem for such systems. We use the function space technique initiated by Balakrishnan [2]. The solution demonstrates that the "separation theorem" holds for time-delayed systems.

2. PROBLEM FORMULATION

We consider the linear stochastic time-delayed system

$$x(t;\omega) = \sum_{i=0}^{k} \int_0^t A_i(s)\, x(s-h_i;\omega)\, ds + \int_0^t B(s)\, u(s;\omega)\, ds + \int_0^t F(s)\, dW(s;\omega)$$

with (2.1)

$$x(t;\omega) = 0 \qquad \text{for } t \leqslant 0;$$

$$Y(t;\omega) = \sum_{i=0}^{k} \int_0^t C_i(s)\, x(s-h_i;\omega)\, ds + \int_0^t G(s)\, dW(s;\omega); \qquad 0 \leqslant t \leqslant 1. \tag{2.2}$$

Here $x(t;\omega)$ and $Y(t;\omega)$ are n- and m-dimensional "states" and "output" functions, respectively; $u(t;\omega)$ is p-dimensional control function; $W(t;\omega)$ is a q-dimensional Wiener process and $A_i(t)$, $B(t)$, $F(t)$, $C_i(t)$, $G(t)$, $i = 0,\ldots, k$, are appropriate dimensional matrix-valued functions. Assume that these coefficient

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functions are all continuous and $G(t)\,G(t)^* > 0$ on the interval $[0, 1]$ of interest, where "$*$" denotes the transpose. The scalar quantities h_i with $0 = h_0 < h_1 < \cdots < h_k$ are the time delays which occur in the system. We are interested in the class of control functions H such that

(i) $\int_0^1 E\,\| u(t; \omega)\|^2\, dt < \infty$, and

(ii) $u(t; \omega)$ is measurable $\beta_Y(t)$ for each t, where $\beta_Y(t)$ is the smallest σ-algebra generated by $Y(s; \omega)$, $0 \leqslant s \leqslant t$.

The existence of solution of (2.1) for $u(t; \omega)$ in H follows from the general result of Viswanathan [3].

We now state the control problem. We want to minimize

$$J(u) = \int_0^1 E([Q(t)\, x(t; \omega), x(t; \omega)] + \lambda[u(t), u(t)])\, dt, \tag{2.3}$$

where $Q(t)$ is continuous in t and is nonnegative definite, and $\lambda > 0$ is fixed. We used $[.,.]$ to denote inner product in two Euclidean spaces of different dimensions. We shall use the same notation to denote the inner product in different Hilbert spaces that will concern us and the space in which the inner product notation is used will be apparent from the context. We shall solve the optimization problem in a subclass H_u of H but that subclass will include all possible controls that are linear functionals of the output data.

3. Specification of H_u

We first perform some elementary transformations and restate results from filtering for time-delayed systems. From (2.1), we can write

$$x(t; \omega) = \int_0^t \Phi(t, \tau)\, B(\tau)\, u(\tau; \omega)\, d\tau + \int_0^t \Phi(t, \tau)\, F(\tau)\, dW(\tau; \omega), \tag{3.1}$$

where the matrix $\Phi(t, \tau)$ satisfies

$$\begin{aligned} &\frac{d}{dt}\, \Phi(t, \tau) = \sum_{i=0}^{k} A_i(t)\, \Phi(t - h_i, \tau) && \text{for } t \geqslant \tau, \\ &\Phi(\tau, \tau) = I, && \text{where } I \text{ is an identity matrix,} \\ &\Phi(t, \tau) = 0 && \text{for } t < \tau. \end{aligned} \tag{3.2}$$

Note that we can write

$$x(t - h_i; \omega) = \int_0^t \Phi(t - h_i, \tau)\, B(\tau)\, u(\tau; \omega)\, d\tau + \int_0^t \Phi(t - h_i, \tau) F(\tau)\, dW(\tau; \omega). \tag{3.1$'$}$$

Whatever the choice of $u(t; \omega)$ in H, let

$$x_u(t; \omega) = \sum_{i=0}^{k} \int_0^t A_i(s)\, x_u(s - h_i; \omega)\, ds + \int_0^t B(s)\, u(s; \omega)\, ds$$

with $x_u(t; \omega) = 0$ for $t \leqslant 0$;

$$Y_u(t; \omega) = \sum_{i=0}^{k} \int_0^t C_i(s)\, x_u(s - h_i; \omega)\, ds;$$

$$\tilde{Y}(t; \omega) = Y(t; \omega) - Y_u(t; \omega)$$

and let

$$\begin{aligned} \tilde{x}(t; \omega) &= x(t; \omega) - x_u(t; \omega); \\ \hat{\tilde{x}}(t, \theta; \omega) &= E(\tilde{x}(t - \theta; \omega) \mid \beta_{\tilde{Y}}(t)); \\ \hat{x}(t, \theta; \omega) &= \hat{\tilde{x}}(t, \theta; \omega) + x_u(t - \theta; \omega); \end{aligned}$$

where $\beta_{\tilde{Y}}(t)$ is the smallest σ-algebra generated by $\tilde{Y}(s; \omega)$, $0 \leqslant s \leqslant t$. Define the innovation process

$$Z_0(t; \omega) = \tilde{Y}(t; \omega) - \sum_{i=0}^{k} \int_0^t C_i(\sigma)\, \hat{\tilde{x}}(\sigma, h_i; \omega)\, d\sigma \tag{3.3}$$

and $\beta_{Z_0}(t)$ is the smallest σ-algebra generated by $Z_0(s; \omega)$, $0 \leqslant s \leqslant t$. Then from [4],

$$\beta_{\tilde{Y}}(t) \equiv \beta_{Z_0}(t) \qquad \text{and} \qquad \hat{\tilde{x}}(t, \theta; \omega) = \int_0^t K(t, \theta, \tau)\, dZ_0(\tau; \omega), \tag{3.4}$$

where $K(t, \theta, \tau) = \sum_{i=0}^{k} P(\tau, \tau - (t - \theta), h_i)\, C_i(\tau)^* \,(G(\tau)\ \ G(\tau)^*)^{-1}$ with $P(t, \theta_1, \theta_2) = E[(x(t - \theta_1) - \hat{x}(t, \theta_1)(x(t - \theta_2) - \hat{x}(t, \theta_2))^*]$.

Partial differential equations determining $P(t, \theta_1, \theta_2)$ are given in [4]. The evolution equation for $\hat{x}(t, \theta; \omega)$ is given by

$$\hat{\tilde{x}}(t, \theta; \omega) + \int_0^t \frac{\partial \hat{\tilde{x}}(s, \theta; \omega)}{\partial \theta}\, ds = \int_0^t K(s, \theta, s)\, dZ_0(s; \omega)$$

with $\hat{\tilde{x}}(t, 0, \omega) = \sum_{i=0}^{k} A_i(s)\, \hat{\tilde{x}}(s, h_i; \omega)\, ds + \int_0^t K(s, 0, s)\, dZ_0(s; \omega)$. Therefore

$$\hat{x}(t, \theta; \omega) + \int_0^t \frac{\partial \hat{x}(s, \theta; \omega)}{\partial \theta}\, ds = \int_0^t K(s, \theta, s)\, dZ_0(s; \omega), \tag{3.5}$$

$$\begin{aligned} \hat{x}(t, 0; \omega) = \sum_{i=0}^{k} \int_0^t A_i(s)\, \hat{x}(s, h_i; \omega)\, ds &+ \int_0^t K(s, 0, s)\, dZ_0(s; \omega) \\ &+ \int_0^t B(s)\, u(s; \omega)\, ds. \end{aligned} \tag{3.6}$$

DEFINITION 1. Let H_u be the class of controls in H with the property that $\beta_Y(t) = \beta_{Z_0}(t)$, $0 \leqslant t \leqslant 1$.

Remark. It is not clear when H_u equals H. We solve the proposed control problem with the control function restricted to H_u.

LEMMA. *H_u is a Hilbert space with inner product defined by*

$$[u, v] = \int_0^1 E[u(t; \omega), v(t; \omega)]\, dt.$$

The space H_u contains functions of the form

$$u(t; \omega) = \int_0^t k(t, s)\, dY(s; \omega), \tag{3.7a}$$

$$u(t; \omega) = \int_0^t k(t, s)\, dZ_0(s; \omega), \tag{3.7b}$$

where $k(t, s)$ is square integrable on the triangle $0 \leqslant s \leqslant t \leqslant 1$ and one of the forms implies the other.

Proof. The result is an obvious generalization of Theorem 7.1 in [2] and follows from (3.1), (3.3), and (3.4), The proof is, therefore, omitted.

DEFINITION 2. Let H_n denote the space of n-dimensional random functions $z(t; \omega)$ with $\int_0^1 E \| z(t; \omega)\|^2\, dt < \infty$. Further, let L denote the linear transformation from H_u into H_n defined by

$$Lu = v; \qquad v(t; \omega) = \int_0^t \Phi(t, s)\, B(s)\, u(s; \omega)\, ds.$$

The transformation L is clearly bounded. We denote the adjoint (in H_u) of this transformation by L^*.

DEFINITION 3. Let Q be the obvious transformation from H_n into itself defined by

$$Qz = y; \qquad y(t; \omega) = Q(t)\, z(t; \omega).$$

4. OPTIMAL CONTROL STRATEGY

We write

$$\begin{aligned} w(t; \omega) = \tilde{x}(t; \omega) &= \hat{x}(t, 0; \omega) + e(t, 0; \omega) \\ &= \int_0^t K(t, 0, \tau)\, dZ_0(\tau; \omega) + e(t, 0; \omega), \end{aligned} \tag{4.1}$$

where $e(t, 0; \omega) = x(t, 0; \omega) - \hat{x}(t, 0; \omega)$. The functional (2.3) can be written as

$$J(u) = [Q(Lu + w), Lu + w] + \lambda[u, u].$$

This is a quadratic form over H_u and the minimum is attained at the unique point u_0 in H_u given by

$$u_0 = -(1/\lambda)\, L^*(Q(Lu_0 + w)). \tag{4.2}$$

Using the form of w given in (4.1), the fact that $E(e(t + \Delta, 0; \omega) \mid \beta_{Z_0}(t)) = 0$ for $\Delta \geqslant 0$ and the argument similar to that in [2, p. 174], it follows that

$$L^*Q(Lu + w)$$

is given by

$$B(t)^*\, E\left(\int_t^1 \Phi(s, t)^*\, Q(s)\, x(s; \omega)\, ds \mid \beta_{Z_0}(t)\right). \tag{4.3}$$

Now (4.3) can be written as

$$B(t)^*\, \hat{Z}(t; \omega),$$

where $\hat{z}(t; \omega) = E(z(t; \omega) \mid \beta_{Z_0}(t))$ with

$$z(t; \omega) = \int_t^1 \Phi(s, t)^*\, Q(s)\, x(s; \omega)\, ds.$$

From [1], $z(t; \omega)$ satisfies

$$\dot{z}(t; \omega) = -\sum_{i=0}^{k} A_i^*(t + h_i)\, z(t + h_i; \omega) - Q(t)\, x(t; \omega)$$

with (4.4)

$$z(t; \omega) = 0 \qquad \text{for } t \geqslant 1.$$

To determine $\hat{z}(t; \omega)$, we recall the results on linear-quadratic control problem for deterministic time-delayed systems. For that problem, from [1], the optimal feedback control law is given by

$$u_0(t) = -(1/\lambda)\, B(t)^*$$

$$\times \left[P_t(0, 0)\, x(t) + \sum_{i=1}^{k} \int_{t-h_i}^{t} P_t(0, \sigma + h_i - t)\, A_i(\sigma + h_i)\, x(\sigma)\right] d\sigma,$$

where $P_t(\theta, s)$ is given by

$$P_t(\theta, s)^* = P_t(s, \theta), \tag{4.6}$$

$$\begin{aligned} -\frac{\partial}{\partial t}' P_t(0, 0) &= A_0^*(t) P_t(0, 0) + P_t(0, 0) A_0(t) \\ &\quad - P_t(0, 0) B(t) B(t)^* P_t(0, 0)/\lambda + Q(t) \\ &\quad + \sum_{i=1}^{k} A_i^*(t + h_i) P_t(h_i, 0) + \sum_{i=1}^{k} P_t(0, h_i) A_i(t + h_i), \end{aligned} \tag{4.7}$$

$$\begin{aligned} \left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) P_t(0, s) &= [A_0^*(t) - P_t(0, 0) B(t) B(t)^*/\lambda] P_t(0, s) \\ &\quad + \sum_{i=1}^{k} A_i^*(t + h_i) P_t(h_i, s), \end{aligned} \tag{4.8}$$

$$\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial s}\right) P_t(\theta, s) = -P_t(\theta, 0) B(t) B(t)^* P_t(0, s)/\lambda, \tag{4.9}$$

with $P_1(\theta, s) = 0$ for $\theta \geqslant 0$ or $s \geqslant 0$.

We now go back to our original problem.

DEFINITION 4. We define

$$\begin{aligned} \xi(t, \theta; \omega) &= P_t(\theta, 0) \hat{x}(t, 0; \omega) \\ &\quad + \sum_{i=1}^{k} \int_{t-h_i}^{t} P_t(0, \sigma + h_i - t) A_i(\sigma + h_i) \hat{x}(t, t - \sigma; \omega) \, d\sigma, \end{aligned}$$

$$z(t, \theta; \omega) = z(t + \theta; \omega),$$

and

$$\eta(t, \theta; \omega) = z(t, \theta; \omega) - \xi(t, \theta; \omega).$$

THEOREM. *The optimal control strategy for our problem is*

$$u_0(t; \omega) = -(1/\lambda) B(t)^* \xi(t, 0; \omega)$$

Proof. Using (4.9) and (3.5),

$$\begin{aligned} &\left(-\partial_t + \frac{\partial}{\partial \theta} dt\right) [P_t(\theta, \sigma + h_i - t) A_i(\sigma + h_i) \hat{x}(t, t - \sigma; \omega)] \\ &\quad = \left[\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial s}\right) P_t(\theta, \sigma + h_i - t)\right] A_i(\sigma + h_i) \hat{x}(t, t - \sigma; \omega) \end{aligned}$$

$$- P_t(\theta, \sigma + h_i - t)\, A_i(\sigma + h_i) \left[\left(\partial_t + \frac{\partial}{\partial\theta}\, dt\right) \hat{x}(t, t - \sigma; \omega)\right]$$

$$= -(1/\lambda)\, P_t(\theta, 0)\, B(t)\, B(t)^*\, P_t(0, \sigma + h_i - t)\, A_i(\sigma + h_i)\, \hat{x}(t, t - \sigma; \omega)\, dt$$

$$- P_t(\theta, \sigma + h_i - t)\, A_i(\sigma + h_i)\, K(t, t - \sigma, t)\, dZ_0(t; \omega).$$

Let $M(t, \theta) = P_t(\theta, 0)\ K(t, 0, t) + \sum_{j=1}^{k} \int_{t-h_j}^{t} P_t(\theta, \sigma + h_i - t)\ A_i(\sigma + h_j)$ $K(t, t - \sigma, t)\, d\sigma$. Using the preceding result and Eqs. (3.5), (4.6), and (4.8), we get after algebraic manipulation,

$$-\partial_t \eta(t, \theta; \omega) + \frac{\partial}{\partial\theta}\, \eta(t, \theta; \omega)\, dt$$

$$= -(1/\lambda)\, P_t(\theta, 0)\, B(t)\, B(t)^*\, \eta(t, 0; \omega)\, dt$$

$$- P_t(\theta, 0)\, B(t)\, B(t)^*/\lambda\, [z(t, 0; \omega) - \hat{z}(t, 0; \omega)]\, dt + M(t, \theta)\, dZ_0(t; \omega), \tag{4.10}$$

where we used for $u(t; \omega)$ the expression $(-1/\lambda)\, B(t)^*\, \hat{z}(t; \omega)$ obtained from (4.3).

On the other hand, using (4.8) and (3.5),

$$\partial_t[P_t(0, \sigma + h_i - t)\, A_i(\sigma + h_i)\, \hat{x}(t, t - \sigma; \omega)]$$

$$= - \left[\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta}\right) P_t(0, \sigma + h_i - t)\right] A_i(\sigma + h_i)\, \hat{x}(t, t - \sigma; \omega)\, dt$$

$$+ P_t(0, \sigma + h_i - t)\, A_i(\sigma + h_i)\left[\left(\partial_t + \frac{\partial}{\partial\theta}\, dt\right) \hat{x}(t, t - \sigma; \omega)\right]$$

$$= \Big[(-A_0^*(t) - P_t(0, 0)\, B(t)\, B(t)^*/\lambda)\, P_t(0, \sigma + h_i - t)$$

$$+ \sum_{j=1}^{k} A_j^*(t + h_j, \sigma + h_i - t)\Big]\, A_i(\sigma + h_i)\, \hat{x}(t, t - \sigma; \omega)\, dt$$

$$+ P_t(0, \sigma + h_i - t)\, A_i(\sigma + h_i)\, K(t, t - \sigma, t)\, dZ_0(t; \omega).$$

From the above result and using (4.7), (3.6), and (4.4), we get after algebraic manipulation,

$$d_t \eta(t, 0; \omega)$$

$$- \sum_{i=0}^{k} A_i^*(t + h_i)\, \eta(t, h_i; \omega)\, dt$$

$$+ P_t(0, 0)\, B(t)\, B(t)^*\, \eta(t, 0; \omega)/\lambda - Q(t)\, (x(t; \omega) - \hat{x}(t, 0; \omega))\, dt$$

$$- \left(\frac{1}{\lambda}\right) P_t(0, 0)\, B(t)\, B(t)^*\, (z(t, 0; \omega) - \hat{z}(t, 0; \omega))\, dt + M(t, 0)\, dZ_0(t; \omega), \tag{4.11}$$

where, as before, we used $-(1/\lambda)\, B(t)^*\, \hat{z}(t; \omega)$ for $u(t; \omega)$. We have the obvious terminal condition

$$\eta(1, \theta; \omega) = 0 \qquad \text{for} \quad \theta \geqslant 0. \tag{4.12}$$

For $\tau < t$, define

$$\hat{\eta}(t, \theta; \omega \mid \tau) = E(\eta(t, \theta; \omega) \mid \beta_{Z_0}(\tau)).$$

Using the fact that $Z_0(t; \omega)$ is a Brownian motion and using properties of stochastic integrals, we get

$$\begin{aligned} &-\partial_t \hat{\eta}(t, \theta; \omega \mid \tau) + \frac{\partial}{\partial \theta} \hat{\eta}(t, \theta; \omega \mid \tau)\, dt \\ &\quad = -(1/\lambda)\, P_t(\theta, 0)\, B(t)\, B(t)^*\, \hat{\eta}(t, 0; \omega \mid \tau)\, dt, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\partial_t \hat{\eta}(t, 0; \omega \mid \tau) \\ &\quad = -\sum_{i=0}^{k} A_i^*(t + h_i)\, \hat{\eta}(t, h_i; \omega \mid \tau)\, dt + (1/\lambda)\, P_t(0, 0)\, B(t)\, B(t)^*\, \hat{\eta}(t, 0; \omega \mid \tau)\, dt, \end{aligned} \tag{4.11}$$

$$\eta(1, \theta; \omega \mid \tau) = 0 \qquad \text{for all} \quad \theta \geqslant 0. \tag{4.12}$$

From uniqueness of solution of (4.10)–(4.12), we get

$$\hat{\eta}(t, \theta; \omega \mid \tau) \equiv 0 \qquad \text{for} \quad \tau < t.$$

Now $\hat{\eta}(t, \theta; \omega \mid \tau)$ being a martingale in τ for fixed t and θ, we have from Doob [5, Theorem 4.3, p. 355]

$$\lim_{\tau \to t-} \hat{\eta}(t, \theta; \omega \mid \tau) = \hat{\eta}(t, \theta; \omega \mid t) \qquad \text{a.e.}$$

so that $\hat{\eta}(t, \theta; \omega \mid t) = 0$. This gives us, in particular for $\theta = 0$,

$$\hat{z}(t; \omega) = \xi(t, 0; \omega)$$

and the theorem is proved.

5. Conclusion

A direct solution has been given for the control problem of linear stochastic time delayed systems with quadratic cost functional. This has enabled us to demonstrate that the "separation theorem" holds for time-delayed systems. The steady-state control for time-independent system matrices is a useful area of investigation.

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